Where has all my sand gone?

Hydro-morphodynamics 2D modelling using a discontinuous Galerkin discretisation

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EPSRC Engineering and Physical Sciences Research Council

Mathematics of Planet Earth

Imperial College London

Overview

Introduction

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February 2014 in Dawlish, Devon

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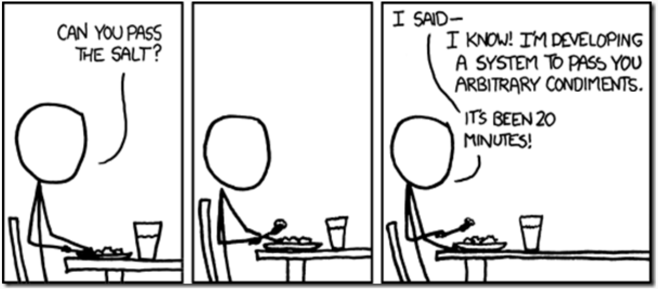
This cost £35 million to fix and is estimated to have cost the Cornish economy £1.2 billion

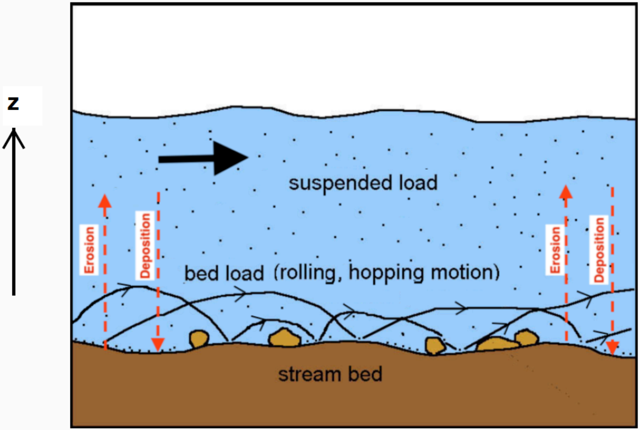
Introduction

Overengineering...



Building a hydro-morphodynamics 2D model in *Thetis*

Sediment Transport



Adapted from `http://geologycafe.com/class/chapter11.html`

Basic Model Equations

Depth-averaging from the bed to the water-surface and filtering turbulence:

Hydrodynamics

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hU_1) + \frac{\partial}{\partial y}(hU_2) = 0, \quad (1)$$

$$\frac{\partial (hU_i)}{\partial t} + \frac{\partial (hU_iU_1)}{\partial x} + \frac{\partial (hU_iU_2)}{\partial y} = -gh\frac{\partial z_s}{\partial x_i} + \frac{1}{\rho}\frac{\partial (hT_{i1})}{\partial x} + \frac{1}{\rho}\frac{\partial (hT_{i2})}{\partial y} - \frac{\tau_{bi}}{\rho}, \quad (2)$$

Basic Model Equations

Depth-averaging from the bed to the water-surface and filtering turbulence:

Conservation of suspended sediment

$$\frac{\partial}{\partial t}(hC) + \frac{\partial}{\partial x}(hF_{corr}U_1C) + \frac{\partial}{\partial y}(hF_{corr}U_2C) = \frac{\partial}{\partial x}\left[h\left(\epsilon_s\frac{\partial C}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[h\left(\epsilon_s\frac{\partial C}{\partial y}\right)\right] + E_b - D_b, \quad (1)$$

where z_s is the fluid surface, τ_{bi} the bed shear stress, T_{ij} the depth-averaged stresses, ϵ_s the diffusivity constant and F_{corr} the correction factor.

Calculating the New Bedlevel

Bedlevel (z_b) is governed by the Exner equation

$$\frac{(1-p')}{m}\frac{dz_b}{dt} + \nabla_h \cdot \mathbf{Q_b} = D_b - E_b, \tag{2}$$

where:

Q_b is the bedload transport given by Meyer-Peter-Müller formula,

$D_b - E_b$ accounts for effects of suspended sediment flow,

m is a morphological factor accelerating bedlevel changes.

Adding Physical Effects

Slope Effect

Accounts for gravity which means sediment moves slower uphill than downhill. We impose a magnitude correction:

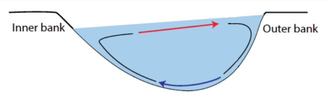
$$\mathbf{Q}_{\mathbf{b}*} = \mathbf{Q}_{\mathbf{b}} \left(1 - \Upsilon \frac{\partial z_b}{\partial s}\right),$$

and a correction on the flow direction (where δ is the original angle)

$$\tan \alpha = \tan \delta - T \frac{\partial z_b}{\partial n}.$$

Secondary Current

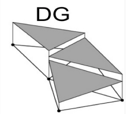
Accounts for the helical flow effect in curved channels



Comparing with Industry Standard Model

Thetis

DG finite element discretisation with
$$\mathrm{P}_{1\mathrm{DG}} - \mathrm{P}_{1\mathrm{DG}}$$



- + Locally mass conservative
- + Well-suited to advection dominated problems
- + Geometrically flexible
- + Allow higher order local approximations

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Thetis

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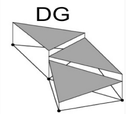

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Telemac-Mascaret

CG finite element discretisation

Method of characteristics (hydrodynamics advection)

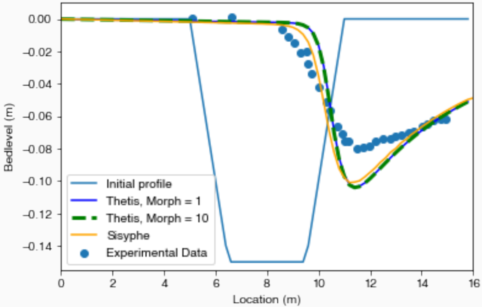
- + Unconditionally stable
- - Not mass conservative
- - Diffusive for small timesteps

Distributive schemes (sediment transport advection)

- + Mass conservative
- - Diffusive for small timesteps
- - Courant number limitations to ensure stability

Migrating Trench

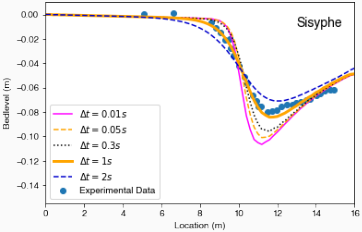
Migrating Trench: Initial Set-up



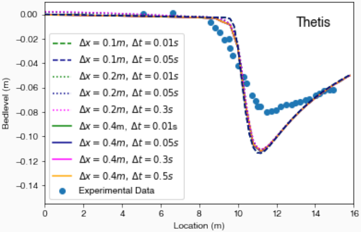
Bedlevel after 15 h *for different morphological scale factors comparing experimental data,* Sisyphe *and* Thetis *with* $\Delta t = 0.05$ s. *Experimental data and initial trench profile source: Villaret et al. (2016)*

Migrating Trench: Issues with Sisyphe

Varying Δt



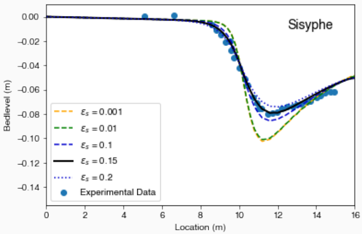
Sisyphe greatly altered by changes to Δt



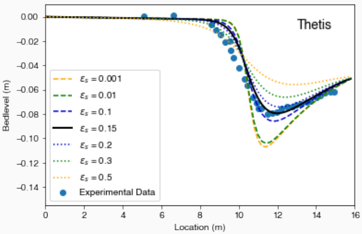
Thetis insensitive to changes in Δt

Migrating Trench: Varying Diffusivity

$$\frac{\partial}{\partial t}(hC) + \frac{\partial}{\partial x}(hF_{corr}U_1C) + \frac{\partial}{\partial y}(hF_{corr}U_2C) = \frac{\partial}{\partial x}\left[h\left(\epsilon_s\frac{\partial C}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[h\left(\epsilon_s\frac{\partial C}{\partial y}\right)\right] + E_b - D_b, \quad (3)$$

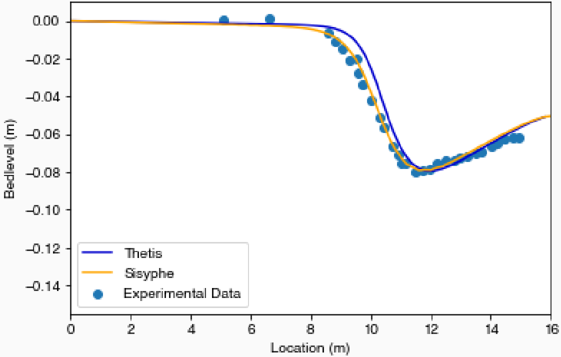


Sensitivity of Sisyphe to ϵ_s



Sensitivity of Thetis to ϵ_s

Migrating Trench: Final Result

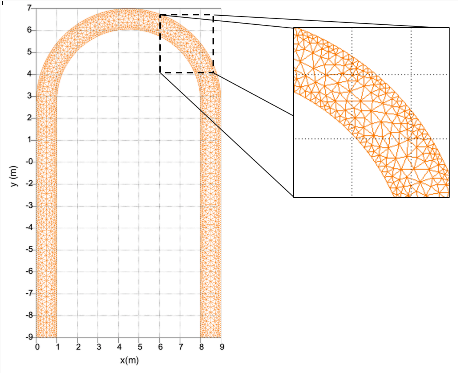


Bedlevel from Thetis and Sisyphe after 15 h

Migrating Trench: Simulation

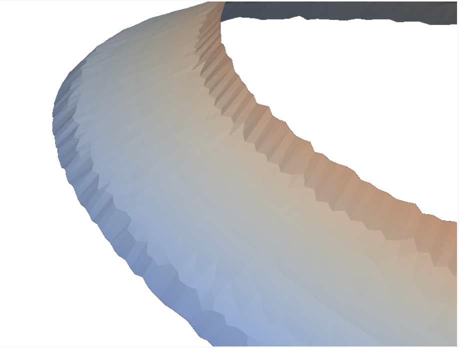
Meander

Meander: Initial Set-up



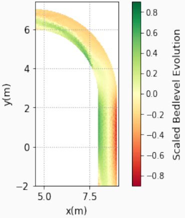
Meander mesh and domain

Meander: Boundary Issue

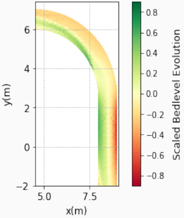


Issue in velocity resolution at boundary resolved by increasing viscosity

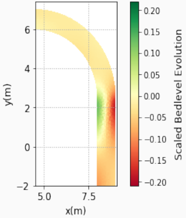
Meander: Physical Effects



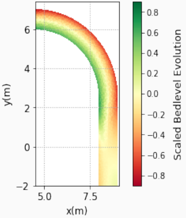
No physical corrections



Only slope effect magnitude

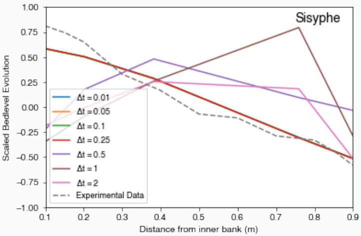


Both slope effect corrections

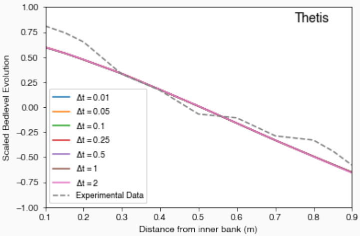


All physical corrections

Meander: Sensitivity to Δt

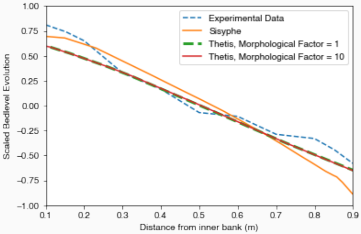


Sisyphe sensitive to changes in Δt

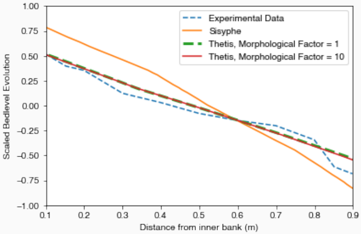


Thetis insensitive to changes in Δt

Meander: Final Result



Cross-section at 90°



Cross-section at 180°

Comparing scaled bedlevel evolution from Thetis, Sisyphe and experimental data

Meander: Simulation

Comparing computational time

	Sisyphe	*Thetis*	*Thetis* (morphological scale factor)	*Thetis* (morphological scale factor, increased Δt)
Migrating Trench	3,427	341,717	39,955	12,422
Meander	980	60,784	10,811	1,212

Comparison of computational time (seconds). For the migrating trench, $\Delta t = 0.05\,\text{s}$ *and increased* $\Delta t = 0.3\,\text{s}$*; for the meander* $\Delta t = 0.1\,\text{s}$ *and increased* $\Delta t = 10\,\text{s}$*.*

Conclusion

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3. Validated our model for two different test cases;
4. Shown our model is both accurate and stable, and has key advantages in robustness and accuracy over the state-of-the-art industry standard Siyphe whilst still being comparable in computational cost

Key References

- Kärnä, T., Kramer, S.C., Mitchell, L., Ham, D.A., Piggott, M.D. and Baptista, A.M. (2018), 'Thetis coastal ocean model: discontinuous Galerkin discretization for the threedimensional hydrostatic equations', *Geoscientific Model Development*, **11**, 4359-4382.
- Tassi, P. and Villaret, C. (2014), *Sisyphe v6.3 User's Manual*, EDF R&D, Chatou, France. Available at: `http://www.opentelemac.org/downloads/MANUALS/SISYPHE/sisyphe`
- Villaret, C., Kopmann, R., Wyncoll, D., Riehme, J., Merkel, U. and Naumann, U. (2016), 'First-order uncertainty analysis using Algorithmic Differentiation of morphodynamic models', *Computers & Geosciences*, **90**, 144-151.
- Villaret, C., Hervouet, J.-M., Kopmann, R., Merkel, U., and Davies, A. G. (2013), 'Morphodynamic modeling using the telemac finite-element system,'*Com- puters & Geosciences*, **53**, 105-113.

Questions?

Using DG:

- Generate a mesh of elements over domain Ω
- Define finite element space on a triangulation (a set of triangles which do not overlap and the union of which is equal to the closure of Ω)
- Derive the weak form of the equation on each triangular element by multiplying the equation by a test function and integrating it by parts on each element and using divergence theorem

Using a discontinuous function space requires the definition of the variables on the element edges thus we use the average and jump operators

$$\{\{\mathsf{X}\}\} = \frac{1}{2}(\mathsf{X}^+ + \mathsf{X}^-), \quad [[\chi]]_\mathsf{n} = \chi^+\mathsf{n}^+ + \chi^-\mathsf{n}^-, \quad [[\mathsf{X}]]_\mathsf{n} = \mathsf{X}^+ \cdot \mathsf{n}^+ + \mathsf{X}^- \cdot \mathsf{n}^-. \tag{4}$$

For C, we use an upwinding scheme, so, at each edge, C is chosen to be equal to its upstream value with respect to velocity. Therefore

$$\int_\Omega \psi \mathbf{u} \cdot \nabla_h C dx = -\int_\Omega C \nabla_h \cdot (\mathbf{u}\psi) dx + \int_\Gamma C^{\mathrm{up}} \, [[\psi \mathbf{u}]]_\mathsf{n} ds. \tag{5}$$

Weak form of diffusivity term uses Symmetric Interior Penalty Galerkin (SIPG) stabilisation method, as if not discretisation unstable for elliptic operators

$$\begin{aligned} -\int_\Omega \psi \nabla_h \cdot (\epsilon_s \nabla_h C) dx &= \int_\Omega \epsilon_s (\nabla_h \psi) \cdot (\nabla_h C) dx - \int_\Gamma [[\psi]]_\mathsf{n} \cdot \{\{\epsilon_s \nabla_h C\}\} ds \\ &\quad - \int_\Gamma [[C]]_\mathsf{n} \cdot \{\{\epsilon_s \nabla_h \psi\}\} ds + \int_\Gamma \sigma \{\{\epsilon_s\}\} [[C]]_\mathsf{n} \cdot [[\psi]]_\mathsf{n} ds. \end{aligned} \tag{6}$$