Anisotropic Goal-Oriented Mesh Adaptation in Firedrake

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Firedrake ’19, Durham
Metric-Based Mesh Adaptation.

Riemannian **metric fields** $\mathcal{M} = \{\mathcal{M}(x)\}_{x \in \Omega}$ are SPD $\forall x \in \mathbb{R}^n$.

$\therefore$ Orthogonal eigendecomposition $\mathcal{M}(x) = V \Lambda V^T$.

Steiner ellipses [Barral, 2015]

Resulting mesh [Barral, 2015]

$\mathbb{E}^2 = (\mathbb{R}^2, I_2) \xrightarrow{\mathcal{M}^{1/2}} (0,1) \xrightarrow{\mathcal{M}^{-1/2}} (1,0) \xrightarrow{\mathcal{M}} (\mathbb{R}^2, \mathcal{M})$

Rokos and Gorman [2013]
Consider interpolating $u \approx \mathcal{I}_h u \in \mathbb{P}1$.

It is shown in [Frey and Alauzet, 2005] that

$$\| u - \Pi_h u \|_{L^\infty(K)} \leq \gamma \max_{x \in K} \max_{e \in \partial K} e^T |H(x)| e$$

where $\gamma > 0$ is a constant related to the spatial dimension.

A metric tensor $\mathcal{M} = \{ \mathcal{M}(x) \}_{x \in \Omega}$ may be defined as

$$\mathcal{M}(x) = \frac{\gamma}{\epsilon} |H(x)|,$$

where $\epsilon > 0$ is the tolerated error level.
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Model Validation

Test Case

Point Discharge with Diffusion

Test case taken from TELEMAC-2D Validation Document 7.0 [Riadh et al., 2014].

\[
\begin{align*}
\mathbf{u} \cdot \nabla \phi - \nabla \cdot (\nu \nabla \phi) &= f \\
\nu \mathbf{n} \cdot \nabla \phi |_{\text{walls}} &= 0 \\
\phi |_{\text{inflow}} &= 0
\end{align*}
\]

\[ f = \delta(x - 2, y - 5) \]

(Provided on a 1,024,000 element uniform mesh.)
Point Discharge with Diffusion: Adjoint Problem

\[ J_i(\phi) = \int_{R_i} \phi \, dx \]

\[ R_1 = B_{\frac{1}{2}}((20, 5)) \]

\[ R_2 = B_{\frac{1}{2}}((20, 7.5)) \]

(Adjoint solution for \( J_1 \).

(Adjoint solution for \( J_2 \).

(Presented on a 1,024,000 element uniform mesh.)
### Point Discharge with Diffusion: Convergence

<table>
<thead>
<tr>
<th>Elements</th>
<th>$J_1(\phi)$</th>
<th>$J_1(\phi_h)$</th>
<th>$J_2(\phi)$</th>
<th>$J_2(\phi_h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4,000</td>
<td>0.20757</td>
<td>0.20547</td>
<td>0.08882</td>
<td>0.08901</td>
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<tr>
<td>16,000</td>
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<td>0.16873</td>
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<td>0.62590</td>
<td>0.06924</td>
<td>0.06922</td>
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<tr>
<td>256,000</td>
<td>0.16344</td>
<td>0.16343</td>
<td>0.06959</td>
<td>0.06958</td>
</tr>
<tr>
<td>1,024,000</td>
<td>0.16344</td>
<td>0.16345</td>
<td>0.06959</td>
<td>0.06958</td>
</tr>
</tbody>
</table>

$J_i(\phi)$ : analytical solutions  
$J_i(\phi_h)$ : $\mathbb{P}1$ finite element solutions
Dual Weighted Residual (DWR).

Given a PDE $\Psi(u) = 0$ and its adjoint written in Galerkin forms

$$
\rho(u_h, v) := L(v) - a(u_h, v) = 0, \quad \forall v \in V_h
$$

$$
\rho^*(u_h^*, v) := J(v) - a(v, u_h^*) = 0, \quad \forall v \in V_h
$$

$\implies$ a posteriori error results [Becker and Rannacher, 2001]

$$
J(u) - J(u_h) = \rho(u_h, u^* - u_h^*) + R^{(2)}
$$

$$
J(u) - J(u_h) = \frac{1}{2} \rho(u_h, u^* - u_h^*) + \frac{1}{2} \rho^*(u_h^*, u - u_h) + R^{(3)}
$$

[Remainders $R^{(2)}$ and $R^{(3)}$ depend on errors $u - u_h$ and $u^* - u_h^*$.]
DWR Integration by Parts

\[ J(u) - J(u_h) = \rho(u_h, u^* - u_h^*) + R^{(2)} \]

Applying integration by parts (again) \textit{elementwise}:

\[ |J(u) - J(u_h)|_K \approx |\langle \Psi(u_h), u^* - u_h^* \rangle_K + \langle \psi(u_h), u^* - u_h^* \rangle_{\partial K}|. \]

- \( \Psi(u_h) \) is the strong residual on \( K \);
- \( \psi(u_h) \) embodies flux terms over elemental boundaries.
Isotropic Metric

\[ |J(u) - J(u_h)|_K \approx \eta := |\langle \Psi(u_h), u^* - u_h^* \rangle_K + \langle \psi(u_h), u^* - u_h^* \rangle_{\partial K}|. \]

Isotropc case:

\[ \mathcal{M} = \begin{bmatrix} \Pi_{P1}\eta & 0 \\ 0 & \Pi_{P1}\eta \end{bmatrix}. \]
Isotropic Meshes

Centred receiver (12,246 elements).

Offset receiver (19,399 elements).
A *posteriori* Approach

Motivated by the approach of [Power et al., 2006], consider the interpolation error:

\[
|J(u) - J(u_h)| \approx |\langle \Psi(u_h), u^* - u_h^* \rangle_K + \langle \psi(u_h), u^* - u_h^* \rangle_{\partial K}|
\]

This suggests the node-wise metric,

\[
\mathcal{M} = |\Psi(u_h)||H(u^*)|,
\]

and correspondingly for the adjoint,

\[
\mathcal{M} = |\Psi^*(u_h^*)||H(u)|.
\]
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Goal-Oriented Adaptation

Anisotropic Goal-Oriented Mesh Adaptation

A posteriori Anisotropic Meshes

Centred receiver (16,407 elements).

Offset receiver (9,868 elements).
A priori Approach

Alternative a priori error estimate [Loseille et al., 2010]:

\[ J(u) - J(u_h) = \langle (\Psi_h - \Psi)(u), u^* \rangle + \tilde{R}. \]

Assume we have the conservative form \( \Psi(u) = \nabla \cdot F(u) \), so

\[ J(u) - J(u_h) \approx \langle (F - F_h)(u), \nabla u^* \rangle_\Omega - \langle \hat{n} \cdot (F - F_h(u), u^*) \rangle_{\partial \Omega}. \]

This gives Riemannian metric fields

\[
\mathcal{M}^{\text{volume}} = \sum_{i=1}^{n} |H(\mathcal{F}_i(u))| \left| \frac{\partial u^*}{\partial x_i} \right|,
\]

\[
\mathcal{M}^{\text{surface}} = |u^*| \left| \overline{H} \left( \sum_{i=1}^{n} \mathcal{F}_i(u) \cdot n_i \right) \right|.
\]
A priori Anisotropic Meshes

Centred receiver (44,894 elements).

Offset receiver (29,143 elements).
Meshes from Combined Metrics (Offset Receiver)

- Averaged isotropic (13,980 elements).
- Superposed isotropic (19,588 elements).
- Averaged a posteriori (9,289 elements).
- Superposed a posteriori (14,470 elements).
- Averaged a priori (25,204 elements).
- Superposed a priori (49,793 elements).
Convergence Analysis: Centred Receiver.
Convergence Analysis: Offset Receiver.
Three Dimensions.

Anisotropic mesh resulting from averaging a posteriori metrics.

Uniform mesh (1,920,000 elements).

Adapted mesh (1,766,396 elements).
Outlook.

To appear in proceedings of the 28th International Meshing Roundtable:

Future work:
- Time dependent (tidal) problems.
- Other finite element spaces, e.g. DG.
- Boundary and flux terms for anisotropic methods.
- Realistic desalination application.
References


Metric Combination.

Consider metrics $\mathcal{M}_1$ and $\mathcal{M}_2$. How to combine these in a meaningful way?

**Metric average:**
$\mathcal{M} := \frac{1}{2}(\mathcal{M}_1 + \mathcal{M}_2)$.

**Metric superposition:**
intersection of Steiner ellipses.

Metric superposition [Barral, 2015]
Double $L_2$ projection

We recover $H = \nabla^T \nabla u$ by solving the auxiliary problem

\[
\begin{cases}
H = \nabla^T g \\
g = \nabla u
\end{cases}
\]

as

\[
\int_{\Omega} \tau : H_h \, dx + \int_{\Omega} \text{div}(\tau) : g_h \, dx - \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{\partial\Omega} (g_h)_i \tau_{ij} n_j \, ds = 0, \quad \forall \tau
\]

\[
\int_{\Omega} \psi \cdot g_h \, dx = \int_{\partial\Omega} u_h \psi \cdot \hat{n} \, ds - \int_{\Omega} \text{div}(\psi) u_h \, dx, \quad \forall \psi.
\]
Accounting for Source Term

Forward equation:

\[ \nabla \cdot \mathcal{F}(\phi) = f, \quad \mathcal{F}(\phi) = u\phi - \nu \nabla \phi. \]

\[ \mathcal{M} = |H(\mathcal{F}_1(\phi))| \left| \frac{\partial \phi^*}{\partial x} \right| + |H(\mathcal{F}_2(\phi))| \left| \frac{\partial \phi^*}{\partial y} \right| + |H(f)| |\phi^*|. \]

Adjoint equation:

\[ \nabla \cdot \mathcal{G}(\phi^*) = g, \quad \mathcal{G}(\phi^*) = -u\phi^* - \nu \nabla \phi^*, \]

\[ \mathcal{M} = |H(\mathcal{G}_1(\phi^*))| \left| \frac{\partial \phi}{\partial x} \right| + |H(\mathcal{G}_2(\phi^*))| \left| \frac{\partial \phi}{\partial y} \right| + |H(g)| |\phi|, \]